EXACT BAYESIAN PREDICTION OF ORDER STATISTICS FROM UNIFORM DISTRIBUTION

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Outline

- Order statistics.
- Prediction of future order statistics based on fixed sample size (FSS).
- Bayesian prediction of future order statistics based on a random sample size (RSS).
- Monte Carlo simulations.
- Applications.

Introduction

The prediction of future observations of order statistics has many applications in the applied studies such as, the biological studies, life testing and quality control problems. Prediction problems come up naturally in several real life situations for example, the prediction of rain fall extremes, highest water level of the seas and temperatures. Also, some other applications involving weather, sport data and in economics. Many distributions have been used in prediction problems, Abd Ellah and Sultan (2005) have considered some predictive intervals for the future observations from the exponential distribution by using the Bayesian approach in both fixed sample size (FSS) and random sample size (RSS). Adopting Bayesian approach, Lingappaiah (1986) has predicted the range in a future sample when sample size is a random variable based on ranges in earlier samples using the same line as in Lingappiah (1978). Evans and Nigm (1980) have discussed the prediction intervals for Weibull distribution, Raqab (2001) has discussed optimal perdition of new observation based on the generalized order statistics. Calabria and Pulicini (1994) have considered prediction from the inverse Weibull distribution. Balakrishnan and Lin (2005) have discussed exact inference and prediction for K-sample exponential case under type-II censoring.

The modified maximum likelihood predictors of future order statistics from normal samples have been discussed by Raqab (1997).

Soliman (2000) has illustrated that the prediction of future order observations and has shown how long a sample of units might run until all fail in life testing. He has used Pareto distribution where the first *i* ordered observations have been observed. AL-Hussaini and Jaheen (1996) have predicted Bayesian bounds for the Burr Type XII distribution in the presence of outliers, Al-Hussaini (2003) has derived the necessary Bayesian predictive density function to obtain the bounds of the predictive intervals of future order statistics. He has applied his approach when the underlying population is a finite mixture of general components. Such components include, among others, the Weibull (exponential and Rayleigh as special cases), three-parameter Burr type XII, Pareto, beta, Gompertz and compound Gompertz distributions. Raqab and Madi (2002) have considered Bayesian prediction of the total time on test by using doubly censored Rayleigh data.

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{i:n}$ be the first i order statistics from the uniform distribution with pdf

$$f(x \mid \theta) = \frac{1}{\theta}, \quad 0 < x < \theta, \qquad (1.1)$$

and $X_{i+1:n} \leq X_{i+2:n} \leq \ldots \leq X_{n:n}$ be the remaining $(n-i)$ order statistics from the

same distribution, the joint pdf of $X_{i:n}$ and $X_{j:n}$ is given by :

$$f_{i,j:n}(x,y) = C_{i,j:n}[F(x)]^{i-1}[F(y) - F(x)]^{j-i-1}[1 - F(y)]^{n-j} f(x) f(y), \qquad (1.2)$$

$$-\infty < x < y < \infty,$$

where $f(.)$ and $F(.)$ are respectively, the pdf and cdf of the uniform distribution given in (1.1) and

$$C_{i,j:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}. \qquad (1.3)$$

In the case of FSS, the Bayes predictive density function of

$$y = X_{j:n},\ j = i+1,\ i+2, .., n$$

for given $X = (X_{1:n}, X_{2:n}, \ldots X_{i:n})$ can be written as ,

$$h(y \mid X) = \int f(y \mid \theta)\Pi(\theta \mid X)d\theta, \qquad (1.4)$$

where $f(y/.)$ is the conditional pdf of the future observation y and $\Pi(\theta/X)$ is the posterior pdf.

In the case of RSS, the predictive distribution function of y when the sample size n is a random variable is given by [see Gupta and Gupta (1984)].

$$q(y \mid X, n) = \frac{1}{Pr(n \geq j)} \sum_{n=j}^{\infty} r(n) h(y \mid X), \qquad (1.5)$$

where $r(n)$ is the probability mass function (pmf) of n and $h(y/X)$ is given in (1.4).

Sultan and Abd Ellah (2006) have used (1.5) to predict the future order statistics from the exponential distribution.

2- Prediction for Fixed sample size

In this section, we use (1.4) to derive the predictive distribution function of the future order statistics from the uniform distribution based on the following statistic

$$W = X_{j:n} - X_{i:n}, \quad 1 \leq i < j \leq n. \tag{2.1}$$

Obtaining the posterior distribution requires the likelihood function. In our case, to predict the *j*-th order statistic ($i \leq j \leq n$), the likelihood function is

$$L(X \mid \theta) = \frac{n!}{(n-i)!}[1 - F(x_i)]^{n-i} \prod_{r=1}^{i} f(x_r), \; 0 < x < \theta. \tag{2.2}$$

Theorem (1) :

Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{i:n}$ be order statistics from uniform distribution $U(0,\theta)$. Considering the prior $g(\theta) = 1,\ 0 < \theta < 1$, then the predictive density function of W in this case can be obtained from (1.4) as

$$h(w \mid x) = \begin{cases} C_{i,j:n} \sum_{r=0}^{n-j} \sum_{k=0}^{n-i} \sum_{d=0}^{n-j+i} (-1)^{d+k+r} \binom{n-j}{r} \binom{n-i}{k} \binom{n-j+i}{d} [(\frac{1}{x_i})^{d+j-1} - x_i^k] \\ \times \frac{w^{d+j-i-1}}{(i+r)(a)(d+j+k-1)},\ 0 < w < x_i \\ \\ C_{i,j:n} \sum_{r=0}^{n-j} \sum_{k=0}^{n-i} \sum_{d=0}^{n-j+i} (-1)^{d+k+r} \binom{n-j}{r} \binom{n-i}{k} \binom{n-j+i}{d} \frac{x_i^k}{(i+r)(a)(d+j+k-1)} \\ \times [(\frac{1}{w})^{i+k} - w^{j+d-i-1}],\ x_i < w < 1, \end{cases} \tag{2.3}$$

where $C_{i,j:n}$ is givenin(1.3).

Lemma(1) :

The corresponding cdf of the predictive pdf in Theorem (1)is given by

$$H(w \mid t) = \begin{cases} \frac{C_{i,j:n}}{a} \sum_{r=0}^{n-j} \sum_{k=0}^{n-i} \sum_{d=0}^{n-j+i} (-1)^{d+k+r} \binom{n-j}{r} \binom{n-i}{k} \binom{n-j+i}{d} \frac{x_i^k}{(i+r)(j+d-i)} \\ \times \frac{[(\frac{1}{x_i})^{d+j+k-1} - 1]}{d+j+k-1} w^{j+d-i}, \ 0 < w < x_i, \\ \\ \frac{C_{i,j:n}}{a} \sum_{r=0}^{n-j} \sum_{k=0}^{n-i} \sum_{d=0}^{n-j+i} (-1)^{d+k+r} \binom{n-j}{r} \binom{n-i}{k} \binom{n-j+i}{d} \\ \times \frac{x_i^k}{(i+r)(d+j+k-1)} \Big[(\frac{1}{k+i-1}) [(\frac{1}{w})^{k+i-1} - (\frac{1}{x_i})^{k+i-1}] \\ \\ -(\frac{w^{j-i+d}}{j-i+d} - \frac{x_i^{j-i+d}}{j-i+d}) + ((\frac{1}{x_i})^{i-1} - x_i^{d+j+k-i}) \Big], \ x_i < w < 1. \end{cases} \tag{2.13}$$

3 Random Samples

In this section, derive the Bayesian predictive distribution of W assuming the random sample size which is binomially distributed with parameter M and p, $n \sim B(n; M, p)$.

Theorem 2

Let $X_{1:n}, \quad \leq \ldots \leq X_{i:n}$ **be the first i order statistics from the uniform distribution and assume that the sample size n is a random variable and is binomially distributed as with pmf**

$$B(n; M, p) = \binom{M}{n} p^n q^{M-n}, \quad q = 1 - p, \quad n = 0, 1, 2, \ldots, M. \tag{3.1}$$

Considering the prior

$$g(\theta) = 1, 0 < \theta < 1$$

and using the predictive density function in (1.5), replacing $r(n)$ by $B(n; M, p)$ given in(3.1), then the predictive density function of W is given by

$$
g(w \mid x) = \begin{cases} \frac{C_{i,j:n}}{1-\sum_{n=0}^{j-1}\binom{M}{n}p^n(1-p)^{M-n}} \sum_{r=0}^{n-j}\sum_{k=0}^{n-i}\sum_{d=0}^{n-j+i}\sum_{n=j}^{M}(-1)^{d+k+r}\binom{M}{n} \\ \times \binom{n-j}{r}\binom{n-i}{k}\binom{n-j+i}{d}\frac{1}{(i+r)(a)(d+j+k-1)} \\ \times p^n(1-p)^{M-n}\left[\left(\frac{1}{x_i}\right)^{d+j-1} - x_i^k\right]w^{d+j-i-1}, \ 0 < w < x_i, \\ \\ \frac{C_{i,j:n}}{1-\sum_{n=0}^{j-1}\binom{M}{n}p^n(1-p)^{M-n}} \sum_{r=0}^{n-j}\sum_{k=0}^{n-i}\sum_{d=0}^{n-j+i}\sum_{n=j}^{M}(-1)^{d+k+r}\binom{M}{n} \\ \times\binom{n-j}{r}\binom{n-i}{k}\binom{n-j+i}{d}p^n(1-p)^{M-n}\frac{x_i^k}{(i+r)(a)(d+j+k-1)} \\ \times \left[\left(\frac{1}{w}\right)^{i+k} - w^{j+d-i-1}\right], \ x_i < w < 1, \end{cases} \tag{3.2}
$$

Lemma(2):

The corresponding cdf of the pdf inTheorem(2)is given by :

$$G(w \mid x)=\begin{cases} \dfrac{C_{i,j:n}}{1-\sum_{n=0}^{j-1}\binom{M}{n}p^n(1-p)^{M-n}}\sum_{r=0}^{n-j}\sum_{k=0}^{n-i}\sum_{d=0}^{n-j+i}\sum_{n=j}^{M}\binom{M}{n}(-1)^{d+k+r}\binom{n-j}{r} \\ \times\binom{n-i}{k}\binom{n-j+i}{d}\dfrac{p^n(1-p)^{M-n}x_i^k w^{j+d-i}}{(i+r)(a)(j+d-i)}\dfrac{[(\frac{1}{x_i})^{d+j+k-1}-1]}{d+j+k-1}, \quad 0<w<x_i \\ \\ \dfrac{C_{i,j:n}}{1-\sum_{n=0}^{j-1}\binom{M}{n}p^n(1-p)^{M-n}}\sum_{r=0}^{n-j}\sum_{k=0}^{n-i}\sum_{d=0}^{n-j+i}\sum_{n=j}^{M}\binom{M}{n}(-1)^{d+k+r}\binom{n-j}{r} \\ \times\binom{n-i}{k}\binom{n-j+i}{d}\dfrac{p^n(1-p)^{M-n}x_i^k}{(i+r)(a)(d+j+k-1)}\Big[(\frac{1}{k+i-1})((\frac{1}{w})^{k+i-1}-(\frac{1}{x_i})^{k+i-1}) \\ -(\frac{w^{j-i+d}}{j-i+d}-\frac{x_i^{j-i+d}}{j-i+d})+((\frac{1}{x_i})^{i-1}-x_i^{d+j+k-i})\Big], \quad x_i<w<1. \end{cases} \qquad (3.4)$$

4 Simulation Results

In this section, we carry out some simulation results to illustrate the usefulness of the theoretical results obtained in Sections (2) and (3).

4 .1 Simulation based on fixed sample size

For the fixed sample size case, we calculate the Bayesian percentage points of the upper bound of the future order statistics as follows:

Algorithm (1):

1. Generate i order statistics $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{i:n}$ from $U(0, \theta)$ by using DRUN from the IMSL.

2. Calculate $F(w)$ at $w = x_{i:n}$ by using (2.13).

3. For a level of significance α (say $\alpha = 0.10$), if $F(x_i) < 1 - \alpha$ then find the percentage point by solving numerically (using routine DZREAL from the IMSL statistical library), $F(w) = 1 - \alpha$, where $F(w)$ is defined by the second part in (2.13) otherwise find the percentage point by solving numerically $F(w) = 1 - \alpha$, where $F(w)$ is defined by the first part in (2.13)

4. Repeat steps 3 and 4 for different values of α and n.

Table(1):The upper percentage point of W based on Bayeian approach

n	i	j	90%	95%	97.5%	99%	n	i	j	90%	95%	97.5%	99%
10	5	6	.12193	.15854	.19520	.24347		16	20	.22400	.25570	.28480	.32070
	5	7	.20591	.25130	.29506	.35036		17	18	.08830	.11330	.13760	.16880
	5	8	.28214	.33405	.38272	.44224		17	19	.14730	.17660	.20420	.23840
	5	9	.35491	.41212	.46427	.52612		17	20	.19970	.23170	.26120	.29720
	5	10	.42575	.48724	.54173	.60437		18	19	.08520	.10930	.13270	.16270
	6	7	.14316	.18329	.22230	.27217		18	20	.14210	.17030	.19680	.22970
	6	8	.23828	.28527	.32932	.38380		19	20	.08480	.10870	.13190	.16150
	6	9	.32266	.37404	.42111	.47780	30	25	26	.06387	.08205	.10017	.18536
	6	10	.40172	.45619	.50501	.56234		25	27	.11976	.11976	.11976	.11976
	7	8	.15181	.19323	.23300	.28324		25	28	.14544	.16971	.19199	.27631
	7	9	.25123	.29854	.34232	.39598		25	29	.07031	.07031	.07031	.07031
	7	10	.33849	.38908	.43499	.49004		25	30	.21621	.24360	.26854	.29834
	8	9	.15332	.19463	.23407	.28362		26	27	.07168	.07168	.07168	.07168
	8	10	.25303	.29964	.34256	.39494		26	28	.11565	.13918	.16137	.18916
	9	10	.14020	.17786	.21384	.25916		26	29	.07188	.07188	.07188	.07188
20	15	16	.08330	.10720	.13050	.16070		26	30	.19603	.22371	.24898	.27968
	15	17	.13930	.01620	.01840	.01880		27	28	.07163	.09216	.11222	.13805
	15	18	.18920	.22030	.24930	.28540		27	29	.07089	.07089	.07089	.07089
	15	19	.23590	.26920	.29980	.33700		27	30	.16274	.18947	.21417	.24447
	15	20	.75950	.70280	.68380	.67750		28	29	.07212	.09277	.11296	.13895
	16	17	.07920	.10180	.12400	.15260		28	30	.12063	.14507	.16811	.19686
	16	18	.13230	.15920	.18460	.21650		29	30	.07188	.09247	.11259	.13850
	16	19	.17970	.20930	.23680	.27090							

Algorithm (1) is applied for some levels of significance $10\%, 5\%, 2.5\%, 1\%$ and for sample sizes $n = 10, 20$ and 30. Table (1) displays the numerical percentage points of the upper bound of the future order statistics based on W. From Table (1), we see that

1. When i increases for a given value of j and a given confidence level, we get better (sharper) upper bounds which is expected since when we increase i more information is obtained.

2. When the pair (i, j) increases and the confidence level increases, the upper bound of W increases.

Example (1) :

In this example, we generate 7 order statistics from $U(0, \theta)$ when $n = 10$ as follows:

0.072, 0.140, 0.162, 0.192, 0.234, 0.366, 0.466.

To predict, for example, the 90% upper bound of the 9-th order statistic by using the 7-th order statistics, we have

$U_{9:10} = U_{7:10} + W = 0.466 + 0.2512 = 0.7172.$

Similarly, for some different values of α and i.

In order to examine the efficiency of our technique, the probability coverage of the predictive confidence intervals can be calculated through Algorithm (2) below:

Algorithm (2) :

1. Generate n order statistics from $U(0, \theta)$.
2. Predict the j-th order statistic $U_{j:n}$ in view of example (1) as $U_{j:n} = W + U_{i:n}, \; i < j$,

   where W is the percentage point given in Table (1).
3. If the simulated value of j-th ordered statistic is less than the predicted value

   $U_{j:n}$, then $ic = ic + 1$, $ic = 0$ at the initial step.

4 . **Go to step 1**

5 . **Repeat steps 1, 2, 3 and 4 up to 10000 runs.**

6 . **The probability coverage** $= \dfrac{ic}{10000}$

Algorithm (2) is applied for different values of a and n and the numerical results are given in Table (2).

Table(2):Probability coverage of upper bound of W based on Bayeian approach

n	i	j	90%	95%	97.5%	99%	n	i	j	90%	95%	97.5%	99%
10	5	6	.7285	.8250	.8886	.9388		16	20	.6860	.7941	.8625	.9240
	5	7	.6500	.7679	.8471	.9207		17	18	.8414	.9073	.9498	.9757
	5	8	.5753	.7100	.8114	.9010		17	19	.8129	.8907	.9369	.9681
	5	9	.5080	.6555	.7733	.8774		17	20	.7919	.8743	.9238	.9620
	5	10	.4338	.5973	.7274	.8490		18	19	.8255	.8947	.9368	.9696
	6	7	.7981	.8774	.9230	.9604		18	20	.7949	.8771	.9252	.9612
	6	8	.7379	.8353	.8956	.9460		19	20	.8286	.8988	.9419	.9717
	6	9	.6862	.7938	.8664	.9277	30	25	26	.8650	.9253	.9598	.9978
	6	10	.6270	.7521	.8387	.9114		25	27	.8925	.8925	.8925	.8925
	7	8	.8100	.8842	.9334	.9668		25	28	.8345	.9057	.9464	.9957
	7	9	.7634	.8510	.9044	.9510		25	29	.1566	.1566	.1566	.1566
	7	10	.7112	.8163	.8829	.9398		25	30	.8066	.8891	.9378	.9682
	8	9	.8062	.8866	.9316	.9658		26	27	.8976	.8976	.8976	.8976
	8	10	.7575	.8489	.9107	.9531		26	28	.8758	.9372	.9661	.9849
	9	10	.7764	.8579	.9053	.9489		26	29	.3574	.3574	.3574	.3574
20	15	16	.8215	.8929	.9382	.9716		26	30	.8662	.9288	.9605	.9827
	15	17	.7895	.0361	.0417	.0424		27	28	.8893	.9434	.9723	.9886
	15	18	.7600	.8528	.9087	.9517		27	29	.6298	.6298	.6298	.6298
	15	19	.7313	.8299	.8935	.9459		27	30	.8848	.9408	.9692	.9876
	15	20	1.000	1.000	1.000	1.000		28	29	.8963	.9465	.9717	.9884
	16	17	.8136	.8868	.9304	.9633		28	30	.8910	.9451	.9727	.9877
	16	18	.7675	.8528	.9089	.9518		29	30	.8945	.9482	.9719	.9894
	16	19	.7234	.8229	.8866	.9354							

Form Table (2), we see that

1 . The probability coverage values are quite close to the corresponding confidence levels for the cases 90%, 95%, 97.5% and 99%.

2 . For almost all cases in the table, the probability coverage increases when the confidence level increases for any pair (i, j).

4 2.Simulation based on random sample size

In this section, we calculate the Bayesian percentage point of W when n follows Binomial(n; M, p). The routine DZREAL from the IMSL was used for solving the nonlinear equation $G_W(w) = 1 - \alpha$, where $G(w)$ is given in (3.4) when n distributed $B(n;\ M, p)$.

Table (3) displays the Bayesian upper bound of the future order statistics from $U(0, \theta)$ when n follows binomial distribution.

Table (3): The upper percentage point of statistic W when n is $B(n; p, M)$

p	M	Possible values of $n \geq 3$	i	j	90%	95%	97.5%	99%
0.3	10	3	2	3	0.22644	0.29923	0.37164	0.46323
		4	2	3	0.22644	0.29923	0.37164	0.46323
			2	4	0.36872	0.45085	0.52517	0.61105
			3	4	0.16518	0.21960	0.27695	0.35449
		5	2	3	0.22644	0.29923	0.37164	0.46323
			2	4	0.36872	0.45085	0.52517	0.61105
			2	5	0.47257	0.55343	0.62248	0.69814
			3	4	0.16518	0.21960	0.27695	0.35449
			3	5	0.28289	0.35247	0.41934	0.50137
			4	5	0.15759	0.20399	0.25202	0.31753
		6	3	4	0.16518	0.21960	0.27695	0.35449
			3	5	0.28289	0.35247	0.41934	0.50137
			3	6	0.38006	0.45444	0.52149	0.59896
			4	5	0.15759	0.20399	0.25202	0.31753
			4	6	0.26628	0.32585	0.38377	0.45643
			5	6	0.21278	0.26712	0.31828	0.38196
		7	3	4	0.16518	0.21960	0.27695	0.35449
			3	5	0.28289	0.35247	0.41934	0.50137
			3	6	0.38006	0.45444	0.52149	0.59896
			3	7	0.45894	0.53278	0.59642	0.66711
			4	5	0.15759	0.20399	0.25202	0.31753
			4	6	0.26628	0.32585	0.38377	0.45643
			4	7	0.35861	0.42362	0.48331	0.55404
			5	6	0.17853	0.22636	0.27289	0.33325
			5	7	0.29091	0.34659	0.39890	0.46329
			6	7	0.18165	0.22920	0.27441	0.33137

0.3	10	10	5	6	0.12202	0.15730	0.19375	0.24486
			5	7	0.20958	0.25737	0.30578	0.36981
			5	8	0.28991	0.34633	0.40068	0.46822
			5	9	0.36231	0.42288	0.47859	0.54482
			5	10	0.42574	0.48723	0.54175	0.60440
			6	7	0.14888	0.18925	0.22866	0.28018
			6	8	0.24724	0.29593	0.34248	0.40136
			6	9	0.33064	0.38385	0.43306	0.49272
			6	10	0.40173	0.45618	0.50500	0.56231
			7	8	0.15711	0.19892	0.23893	0.28969
			7	9	0.25689	0.30486	0.34948	0.40458
			7	10	0.33849	0.38908	0.43499	0.49003
			8	9	0.15566	0.19703	0.23638	0.28578
			8	10	0.25304	0.29965	0.34256	0.39496
			9	10	0.14020	0.17786	0.21384	0.25917

In order to examine the efficiency of our technique when the sample size is random variable, the probability coverage values of the predictive confidence intervals are calculated when n is binomially distributed based on 10, 000 simulations in Table (4).

Example (2):

To show the usefulness of the Bayesian percentage point when n follows the binomial distribution, we generate the value of n from the binomial generator $B(n; 10, 0.3)$ to get $n = 10$. Next, we generate 6 order statistics when $n = 10$ from $U(0, \theta)$ as:

0.0721, 0.1401, 0.1619, 0.1917, 0.2339, 0.3656.

By using the 6-th order statistic, we predict the Bayesian upper bound of the 7-th order statistic. Referring to Table (3) when $p = 0.3$, $M = 10$, $n = 10$, we have the 90% predictive upper bound of the 7-th order statistic is $U_{7:10} = U_{6:10} + W = 0.3656 + 0.14888 = 0.51448$. and the 95% upper bound is

$X_{7:10} = 0.3656 + 0.18925 = 0.55485$.

Similarly, we can predict the 8-th, 9-th and 10-th order statistics.

Table (4): **The probability coverage**

p	M	i	j	90%	95%	97.5%	99%
0.3	10	5	6	.7285	.8235	.8865	.9406
		5	7	.6603	.7812	.8627	.9386
		5	8	.5980	.7377	.8424	.9270
		5	9	.5278	.6825	.8009	.8999
		5	10	.4338	.5973	.7274	.8491
		6	7	.8122	.8874	.9299	.9643
		6	8	.7583	.8508	.9104	.9572
		6	9	.7045	.8104	.8831	.9417
		6	10	.6270	.7521	.8387	.9114
		7	8	.8228	.8933	.9384	.9691
		7	9	.7755	.8596	.9130	.9565
		7	10	.7112	.8163	.8829	.9398
		8	9	.8130	.8898	.9344	.9665
		8	10	.7575	.8489	.9107	.9531
		9	10	.7764	.8579	.9053	.9489
0.5	10	5	6	.7316	.8257	.8896	.9427
		5	7	.6614	.7817	.8627	.9380
		5	8	.5972	.7367	.8409	.9260
		5	9	.5268	.6812	.7993	.8984
		5	10	.4338	.5973	.7274	.8491
		6	7	.8122	.8878	.9301	.9645
		6	8	.7574	.8498	.9101	.9569
		6	9	.7031	.8101	.8825	.9409
		6	10	.6270	.7521	.8387	.9114
		7	8	.8223	.8930	.9382	.9690
		7	9	.7749	.8591	.9126	.9562
		7	10	.7112	.8162	.8829	.3721
		8	9	.8129	.8897	.9342	.9665
		8	10	.7575	.8489	.9107	.9531
		9	10	.7764	.8579	.9053	.9489

4 3. Applications

The following data given by Proschan (1963) that are the times between successive failure air conditioning equipment in a Boeing 720 airplane, arranged in increasing order of magnitude. The first nine of the data points are: 12, 21, 26, 29, 29, 48, 57, 59, 70. Upon using the percentage points of Statistic in section (2.2), we predict the 90% upper bounds for the 10-th observation based on the first 9 order observations. First, we calculate the maximum likelihood estimate of λ as:

$$\hat{\lambda} = \frac{r}{\sum_r^r t_i + (n-i)t_r} = \frac{9}{351+70} = 0.02137767.$$

Then $U_{9:10} = 0.776073$ and from Table (3), we have $U_{10:10} = 0.776073 + 0.14020 = 0.916273$. Hence, the 90% predictive upper bound of the 10-th order statistic from the exponential distribution is given by:

$$T_{10:10} = \frac{-\ln(1 - x_{10:10})}{\hat{\lambda}} = 116.$$

Remark:

1. We notice from the results that as we increase the value of i for a given value of j and a given confidence level, we get better (sharper) upper bounds which is expected since when we increase i more information is obtained.

2. We also notice that for a fixed pair (i, j), the upper bounds of W increase as the confidence level increases.

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THANK YOU